6.5 Trapezoids and Kites

**GOAL 1** **Using Properties of Trapezoids**

A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the bases. A trapezoid has two pairs of base angles. For instance, in trapezoid $ABCD$, $\angle D$ and $\angle C$ are one pair of base angles. The other pair is $\angle A$ and $\angle B$. The nonparallel sides are the legs of the trapezoid.

If the legs of a trapezoid are congruent, then the trapezoid is an isosceles trapezoid.

You are asked to prove the following theorems in the exercises.

### THEOREMS

**THEOREM 6.14**

If a trapezoid is isosceles, then each pair of base angles is congruent.

$\angle A \cong \angle B$, $\angle C \cong \angle D$

**THEOREM 6.15**

If a trapezoid has a pair of congruent base angles, then it is an isosceles trapezoid.

$ABCD$ is an isosceles trapezoid.

**THEOREM 6.16**

A trapezoid is isosceles if and only if its diagonals are congruent.

$ABCD$ is isosceles if and only if $AC \cong BD$.

---

**EXAMPLE 1** Using Properties of Isosceles Trapezoids

$PQRS$ is an isosceles trapezoid. Find $m\angle P$, $m\angle Q$, and $m\angle R$.

**SOLUTION** $PQRS$ is an isosceles trapezoid, so $m\angle R = m\angle S = 50^\circ$. Because $\angle S$ and $\angle P$ are consecutive interior angles formed by parallel lines, they are supplementary. So, $m\angle P = 180^\circ - 50^\circ = 130^\circ$, and $m\angle Q = m\angle P = 130^\circ$. 

---
Show that $ABCD$ is a trapezoid.

**SOLUTION**

Compare the slopes of opposite sides.

The slope of $AB = \frac{5 - 0}{0 - 5} = \frac{5}{-5} = -1$.

The slope of $CD = \frac{4 - 7}{7 - 4} = \frac{-3}{3} = -1$.

The slopes of $AB$ and $CD$ are equal, so $AB \parallel CD$.

The slope of $BC = \frac{7 - 5}{4 - 0} = \frac{2}{4} = \frac{1}{2}$.

The slope of $AD = \frac{4 - 0}{7 - 5} = \frac{4}{2} = 2$.

The slopes of $BC$ and $AD$ are not equal, so $BC$ is not parallel to $AD$.

So, because $AB \parallel CD$ and $BC$ is not parallel to $AD$, $ABCD$ is a trapezoid.

The **midsegment** of a trapezoid is the segment that connects the midpoints of its legs. Theorem 6.17 is similar to the Midsegment Theorem for triangles. You will justify part of this theorem in Exercise 42. A proof appears on page 839.

**THEOREM**

**THEOREM 6.17 Midsegment Theorem for Trapezoids**

The midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

$MN \parallel AD, MN \parallel BC, MN = \frac{1}{2}(AD + BC)$

**EXAMPLE 3 Finding Midsegment Lengths of Trapezoids**

**LAYER CAKE** A baker is making a cake like the one at the right. The top layer has a diameter of 8 inches and the bottom layer has a diameter of 20 inches. How big should the middle layer be?

**SOLUTION**

Use the Midsegment Theorem for Trapezoids.

$$DG = \frac{1}{2}(EF + CH) = \frac{1}{2}(8 + 20) = 14 \text{ inches}$$
**GOAL 2  USING PROPERTIES OF KITES**

A *kite* is a quadrilateral that has two pairs of consecutive congruent sides, but opposite sides are not congruent. You are asked to prove Theorem 6.18 and Theorem 6.19 in Exercises 46 and 47.

**THEOREMS ABOUT KITES**

**THEOREM 6.18**
If a quadrilateral is a kite, then its diagonals are perpendicular.

**THEOREM 6.19**
If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.

**EXAMPLE 4  Using the Diagonals of a Kite**

WXYZ is a kite so the diagonals are perpendicular. You can use the Pythagorean Theorem to find the side lengths.

\[ WX = \sqrt{20^2 + 12^2} \approx 23.32 \]
\[ XY = \sqrt{12^2 + 12^2} \approx 16.97 \]

Because WXYZ is a kite, \( WZ = WX \approx 23.32 \) and \( ZY = XY \approx 16.97 \).

**EXAMPLE 5  Angles of a Kite**

Find \( m \angle G \) and \( m \angle J \) in the diagram at the right.

**SOLUTION**

GHJK is a kite, so \( \angle G \equiv \angle J \) and \( m \angle G = m \angle J \).

\[ 2(m \angle G) + 132^\circ + 60^\circ = 360^\circ \]
\[ 2(m \angle G) = 168^\circ \]
\[ m \angle G = 84^\circ \]

So, \( m \angle J = m \angle G = 84^\circ \).
1. Name the bases of trapezoid $ABCD$.

2. Explain why a rhombus is not a kite. Use the definition of a kite.

Decide whether the quadrilateral is a trapezoid, an isosceles trapezoid, a kite, or none of these.

3. 

4. 

5. 

6. How can you prove that trapezoid $ABCD$ in Example 2 is isosceles?

Find the length of the midsegment.

7. 

8. 

9. 

STUDYING A TRAPEZOID

Draw a trapezoid $PQRS$ with $QR \parallel PS$. Identify the segments or angles of $PQRS$ as bases, consecutive sides, legs, diagonals, base angles, or opposite angles.

10. $QR$ and $PS$ 

11. $PQ$ and $RS$ 

12. $PQ$ and $QR$ 

13. $QS$ and $PR$ 

14. $\angle Q$ and $\angle S$ 

15. $\angle S$ and $\angle P$

FINDING ANGLE MEASURES 

Find the angle measures of $JKLM$.

16. 

17. 

18. 

FINDING MIDSEGMENTS 

Find the length of the midsegment $MN$.

19. 

20. 

21.
**Chapter 6 Quadrilaterals**

**USING ALGEBRA** Find the value of $x$.

22. 23. 24.

**CONCENTRIC POLYGONS** In the diagram, $ABCDEFGHJKLM$ is a regular dodecagon, $AB \parallel PQ$, and $X$ is equidistant from the vertices of the dodecagon.

25. Are you given enough information to prove that $ABPQ$ is isosceles? Explain your reasoning.

26. What is the measure of $\angle AXB$?

27. What is the measure of each interior angle of $ABPQ$?

**USING ALGEBRA** What are the lengths of the sides of the kite? Give your answer to the nearest hundredth.

28. 29. 30.

**ANGLES OF KITES** $EFGH$ is a kite. What is $m \angle G$?

31. 32. 33.

34. **ERROR ANALYSIS** A student says that parallelogram $ABCD$ is an isosceles trapezoid because $AB \parallel DC$ and $AD \cong BC$. Explain what is wrong with this reasoning.

35. **CRITICAL THINKING** The midsegment of a trapezoid is 5 inches long. What are possible lengths of the bases?

36. **COORDINATE GEOMETRY** Determine whether the points $A(4, 5), B(-3, 3), C(-6, -13),$ and $D(6, -2)$ are the vertices of a kite. Explain your answer.

**TRAPEZOIDS** Determine whether the given points represent the vertices of a trapezoid. If so, is the trapezoid isosceles? Explain your reasoning.

37. $A(-2, 0), B(0, 4), C(5, 4), D(8, 0)$

38. $E(1, 9), F(4, 2), G(5, 2), H(8, 9)$
39. **Layer Cake** The top layer of the cake has a diameter of 10 inches. The bottom layer has a diameter of 22 inches. What is the diameter of the middle layer?


   **Given**
   
   \[AB \parallel DC, \overline{AD} \cong \overline{BC}\]

   **Prove**
   
   \[\angle D \cong \angle C, \angle DAB \cong \angle B\]

   **Plan for proof** To show \(\angle D \cong \angle C\), first draw \(AE \parallel AB\) so \(\triangle ABCE\) is a parallelogram. Then show \(BC \parallel AE\), so \(\overline{AE} \cong \overline{AD}\) and \(\angle D \cong \angle AED\). Finally, show \(\angle D \cong \angle C\). To show \(\angle DAB \cong \angle B\), use the consecutive interior angles theorem and substitution.

41. **Proving Theorem 6.16** Write a proof of one conditional statement of Theorem 6.16.

   **Given**
   
   \[TQRS\] is an isosceles trapezoid.

   \[\overline{QR} \parallel \overline{TS}\] and \(\overline{QT} \cong \overline{RS}\)

   **Prove**
   
   \(\overline{TR} \cong \overline{SQ}\)

42. **Justifying Theorem 6.17** In the diagram below, \(\overline{BG}\) is the midsegment of \(\triangle ACD\) and \(\overline{GE}\) is the midsegment of \(\triangle ADF\). Explain why the midsegment of trapezoid \(ACDF\) is parallel to each base and why its length is one half the sum of the lengths of the bases.

43. **Using Technology** In Exercises 43–45, use geometry software.

   **Draw points** \(A, B, C\) and segments \(\overline{AC}\) and \(\overline{BC}\). Construct a circle with center \(A\) and radius \(AC\). Construct a circle with center \(B\) and radius \(BC\). Label the other intersection of the circles \(D\). Draw \(\overline{BD}\) and \(\overline{AD}\).

   **What kind of shape is** \(ACBD\)? **How do you know?** What happens to the shape as you drag \(A\)? drag \(B\)? drag \(C\)?

   **Measure** \(\angle ACB\) and \(\angle ADB\). **What happens to the angle measures as you drag** \(A, B,\) or \(C\)?

   **Which theorem does this construction illustrate?**
46. **PROVING THEOREM 6.18** Write a two-column proof of Theorem 6.18.

**GIVEN** $\overline{AB} \cong \overline{CB}, \overline{AD} \cong \overline{CD}$

**PROVE** $\overline{AC} \perp \overline{BD}$

47. **PROVING THEOREM 6.19** Write a paragraph proof of Theorem 6.19.

**GIVEN** $ABCD$ is a kite with $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$.

**PROVE** $\angle A \cong \angle C, \angle B \not\cong \angle D$

**Plan for Proof** First show that $\angle A \cong \angle C$. Then use an indirect argument to show $\angle B \cong \angle D$: If $\angle B \cong \angle D$, then $ABCD$ is a parallelogram. But opposite sides of a parallelogram are congruent. This contradicts the definition of a kite.

**TRAPEZOIDS** Decide whether you are given enough information to conclude that $ABCD$ is an isosceles trapezoid. Explain your reasoning.

48. $\overline{AB} \parallel \overline{DC}$

49. $\overline{AB} \parallel \overline{DC}$

50. $\angle A \cong \angle B$

51. **MULTIPLE CHOICE** In the trapezoid at the right, $NP = 15$. What is the value of $x$?

A $\ 2$  
B $\ 3$  
C $\ 4$  
D $\ 5$  
E $\ 6$

52. **MULTIPLE CHOICE** Which one of the following can a trapezoid have?

A congruent bases  
B diagonals that bisect each other  
C exactly two congruent sides  
D a pair of congruent opposite angles  
E exactly three congruent angles

53. **PROOF** Prove one direction of Theorem 6.16: If the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.

**GIVEN** $PQRS$ is a trapezoid.

$QR \parallel PS, PR \equiv SQ$

**PROVE** $QP \equiv RS$

**Plan for Proof** Draw a perpendicular segment from $Q$ to $\overline{PS}$ and label the intersection $M$. Draw a perpendicular segment from $R$ to $\overline{PS}$ and label the intersection $N$. Prove that $\triangle QMS \cong \triangle RNP$. Then prove that $\triangle QPS \cong \triangle RSP$. 

362  Chapter 6  Quadrilaterals
**Mixed Review**

**Conditional Statements**  Rewrite the statement in if-then form.  (Review 2.1)

54. A scalene triangle has no congruent sides.
55. A kite has perpendicular diagonals.
56. A polygon is a pentagon if it has five sides.

**Finding Measurements**  Use the diagram to find the side length or angle measure.  (Review 6.2 for 6.6)

57. $LN$
58. $KL$
59. $ML$
60. $JL$
61. $m\angle JML$
62. $m\angle MJK$

**Parallelograms**  Determine whether the given points represent the vertices of a parallelogram. Explain your answer. (Review 6.3 for 6.6)

63. $A(-2, 8), B(5, 8), C(2, 0), D(-5, 0)$
64. $P(4, -3), Q(9, -1), R(8, -6), S(3, -8)$

**Quiz 2**

1. **Positioning Buttons**  The tool at the right is used to decide where to put buttons on a shirt. The tool is stretched to fit the length of the shirt, and the pointers show where to put the buttons. Why are the pointers always evenly spaced?  (Hint: You can prove that $HI \cong JK$ if you know that $\triangle JFK \cong \triangle HEJ$)  (Lesson 6.4)

Determine whether the given points represent the vertices of a rectangle, a rhombus, a square, a trapezoid, or a kite.  (Lessons 6.4, 6.5)

2. $P(2, 5), Q(-4, 5), R(2, -7), S(-4, -7)$
3. $A(-3, 6), B(0, 9), C(3, 6), D(0, -10)$
4. $J(-5, 6), K(-4, -2), L(4, -1), M(3, 7)$
5. $P(-5, -3), Q(1, -2), R(6, 3), S(7, 9)$
6. **Proving Theorem 6.15**  Write a proof of Theorem 6.15.

**Given**  $ABCD$ is a trapezoid with $AB \parallel DC$.

**Prove**  $AD \parallel BC$

**Plan for Proof**  Draw $\overline{AE}$ so $\overline{ABCE}$ is a parallelogram. Use the Transitive Property of Congruence to show $\angle AED \equiv \angle D$. Then $\overline{AD} \equiv \overline{AE}$, so $\overline{AD} \equiv \overline{BC}$.  (Lesson 6.5)